ON THE RELATION BETWEEN SCHWARZSCHILD'S AND KERR'S MANIFOLDS

ANGELO LOINGER AND TIZIANA MARSICO

Abstract. Kerr's manifold is only a Schwarzschild's manifold as "seen" by a suitably rotating coordinate system. By taking into account this fact, Kerr's manifold can be "reduced" to a Schwarzschild's manifold. – In a final *aperçu* we summarize the main steps of our reasoning.

PACS 04.20 – General relativity.

1. – The standard (Hilbert-Droste-Weyl) form for the $\mathrm{d}s^2$ of Schwarzschild's manifold of a gravitating point mass m is – if r', ϑ', φ' are spherical polar coordinates:

$$\begin{aligned} \mathrm{d}s^2 \quad &= \quad \left(1-\frac{2m}{r'}\right)^{-1}\mathrm{d}r'^2 + r'^2\left(\mathrm{d}\vartheta'^2+\sin^2\vartheta'\mathrm{d}\varphi'^2\right) - \\ &\quad -\left(1-\frac{2m}{r'}\right)\mathrm{d}t'^2 \quad ; \quad (G=c=1) \quad . \end{aligned} \tag{1}$$

With Boyer and Lindquist (see [1], [2]), the $\mathrm{d}s^2$ of Kerr's manifold can be written as follows:

$$\begin{aligned} \mathrm{d}s^2 \quad &= \quad \left(\frac{\mathrm{d}r^2}{\Delta}+\mathrm{d}\vartheta^2\right)\Sigma + (r^2+a^2)\sin^2\vartheta\,\mathrm{d}\varphi^2 - \mathrm{d}t^2 + \\ &\quad + \frac{2mr}{\Sigma}\left(a\sin^2\vartheta\,\mathrm{d}\varphi+\mathrm{d}t\right)^2 \quad ; \quad (G=c=1) \quad , \end{aligned} \tag{2}$$

where: $\Sigma \equiv r^2+a^2\cos^2\vartheta$; $\Delta \equiv r^2-2mr+a^2$. The parameter a has a geometrical and kinematical meaning. *The case $m \geq a$ is physically interesting.* When $a=0$, eq. (2) coincides with eq. (1).

2. – The potential g_{jk}, $(j,k=1,2,3,4)$, of eq. (2) is referred to a frame which rotates with the following angular velocity $g_{t\varphi}/g_{\varphi\varphi} \equiv \omega$:

$$\omega = \frac{2\,m\,a\,r}{(r^2+a^2)(r^2+a^2\cos^2\vartheta)+2\,m\,a^2 r\sin^2\vartheta} \quad ; \tag{2'}$$

at $r=0$, we have $\omega=0$: this means, strictly speaking, that the gravitating point mass does *not* rotate. For $r \neq 0$ and $m \neq 0$, ω is equal to

zero if and only if the parameter a is equal to zero, and in this case Kerr's manifold coincides with Schwarzschild's manifold, as we have seen in sect.**1**. – Of course, an $\omega \neq 0$ generates *dragging* forces.

3. – Kerr's surface $r = m + (m^2 - a^2 \cos^2 \vartheta)^{1/2}$, which is tangent to the surface $r = m + (m^2 - a^2)^{1/2}$ at $\vartheta = 0$ and $\vartheta = \pi$, has this property: if we make in eq. (2) the coordinate shift (a legitimate choice of a new radial coordinate):

$$r \to r + m + (m^2 - a^2 \cos^2 \vartheta)^{1/2} \quad , \tag{3}$$

we obtain a $\Delta \geq 0$; the equality holds for $r = 0$ and $\cos^2 \vartheta = 1$. Indeed, the new Δ is:

$$\begin{aligned} \Delta \quad = \quad & \left[r + (m^2 - a^2 \cos^2 \vartheta)^{1/2} - (m^2 - a^2)^{1/2}\right] \cdot \\ & \cdot \left[r + (m^2 - a^2 \cos^2 \vartheta)^{1/2} + (m^2 - a^2)^{1/2}\right] \geq 0 \quad ; \end{aligned} \tag{4}$$

and if $\cos^2 \vartheta = 1$:

$$\Delta_{\vartheta=0,\pi} = r \cdot \left[r + 2\,(m^2 - a^2)^{1/2}\right] \quad . \tag{4'}$$

When $a = 0$, transformation (3) becomes:

$$r \to r + 2m \quad , \tag{3'}$$

which coincides with Brillouin (-Schwarzschild) transformation of radial coordinate r' of eq. (1) [3].

Brillouin's form of Schwarzschildian $\mathrm{d}s^2$ and the above new form of Kerr's $\mathrm{d}s^2$ have *both* a ***sole*** (and "soft") singularity at $r' = r = 0$. This means that eq. (1) and eq. (2) have a physical (and mathematical) meaning only when $r' > 2m$ and $r > m + (m^2 - a^2 \cos^2 \vartheta)^{1/2}$, respectively. Our paper quoted in [2] gives a striking proof of this assertion. (Accordingly, there is no room for the physical existence of BH's). – Remark that the new form (*à la* Brillouin) of Kerr's metric is ***diffeomorphic*** to the *exterior* part (*i.e.*, for $r > m + (m^2 - a^2 \cos^2 \vartheta)^{1/2}$) of the form of eq. (2), and is *maximally extended*. All the *observational* results concern only the exterior part of eq. (2), or equivalently the $r > 0$ region of the new form of $\mathrm{d}s^2$.

4. – We have seen that the singular surfaces $r' = 2m$ and $r = m + (m^2 - a^2 \cos^2 \vartheta)^{1/2}$ are in a strict correspondence. Moreover, we prove that this *Kerr's singular surface can be transformed into the surface $r' = 2m$ with an appropriate change of general coordinates.*

From the standpoint of a three-dimensional Euclidean *Bildraum*, a "vertical" section of the surface $r = m + (m^2 - a^2)^{1/2}$ is an ellipse (see Appendix);

accordingly, this surface is a rotational ellipsoid, i.e. an oblate spheroid. And the surface $r' = 2m$ is a sphere.

The semi-axes, say α and β, of the above ellipse are:

$$\begin{cases} \alpha = 2m \quad ; \quad \beta = m + (m^2 - a^2)^{1/2} \quad ; \quad \Rightarrow \\ \alpha^2 - \beta^2 = a^2 + 2m \left[m - (m^2 - a^2)^{1/2} \right] \quad . \end{cases} \tag{5}$$

We remark that

$$\{a = 0\} \Leftrightarrow \{\gamma^2 \equiv \alpha^2 - \beta^2 = 0\} \quad . \tag{6}$$

Obviously, if ξ, η are Cartesian orthogonal coordinates, the equation of our ellipse can be also written as follows:

$$\frac{\xi^2}{\alpha^2} + \frac{\eta^2}{\beta^2} = 1 \quad . \tag{7}$$

If $\xi = \xi'$, $\eta = (\beta/\alpha)\,\eta'$, we have

$$\xi'^{\,2} + \eta'^{\,2} = (2m)^2 \quad , \tag{8}$$

where:

$$\frac{\beta^2}{\alpha^2} = \frac{2\,m\,[\,m + (m^2 - a^2)^{1/2}] - a^2}{4\,m^2} \quad . \tag{9}$$

The transformation of the spheroid $r = m + (m^2 - a^2 \cos^2\vartheta)^{1/2}$ into the sphere $r' = 2m$ is an immediate consequence of eq. (8).

This result has been obtained with a simple application of the following theorem of projective geometry: if we have an ellipsoid (resp. an ellipse) and a sphere (resp. a circle), there exist *collineations* that transform the ellipsoid (resp. the ellipse) into the sphere (resp. the circle).

The parameter a (its geometrical meaning is explained by eq. (6)), which is responsible for the spinning of Kerr's frame, has been "incorporated" in new coordinates; this implies that Kerr's manifold is *not* substantially distinct from Schwarzschild's manifold, because *the "soft" singularities $r' = 2m$ and $r = m + (m^2 - a^2 \cos^2\vartheta)^{1/2}$ – which represent projectively the **same** geometrical object – characterize completely these manifolds.* Indeed, in the shifted coordinates *à la* Brillouin (-Schwarzschild) both manifolds are solutions of Einstein equations $R_{jk} = 0$, $(j, k = 1, 2, 3, 4)$, with *one and only* "soft" singularity at the origin of the radial coordinates ($r' = r = 0$); the surfaces $r = 2m$ and $r = m + (m^2 - a^2 \cos^2\vartheta)^{1/2}$ are really *not* surfaces but *single points.*

5. – The above conclusion is similar to this Weyl's result [4]: the $\mathrm{d}s^2$ of eq. (1) can be expressed also in a cylindrical system of coordinates (Weyl's "canonical" system) z^*, r^*, ϑ^*. Then, the "globe" $r' = 2m$ becomes the "segment" $-m \le z^* \le +m$.

Our case is a little more involved, owing to the presence of the parameter a, which however can be "taken up" by the coordinate change that allows the transformation of the spheroid $r = m + (m^2 - a^2 \cos^2 \vartheta)^{1/2}$ into the sphere $r' = 2m$.

When $m = 0$ and $a \neq 0$, there is a simple relation between $(r', \vartheta', \varphi', t')$ and $(r, \vartheta, \varphi, t)$, see Appendix of paper [2]; in this case, eq. (1) and eq. (2) give only two different forms of *Minkowski* interval $\mathrm{d}s_M^2$.

6. – A consideration on the role of the *Killing vectors* [5]. As it is well known, they yield an *invariant* description of the symmetry properties of a given manifold. However, it is necessary to distinguish the Riemann-Einstein manifolds generated by *extended* material distributions from the Riemann-Einstein manifolds generated by *punctual* masses, that are solutions of $R_{jk} = 0$ with a singularity at the origin of the coordinates. In this second case, we can have manifolds with *any* kind of symmetry: indeed, a mass point can be considered as a kind of limit of a material distribution of any symmetry – and therefore a coordinate system "adapted" to a chosen symmetry is also adequate to the field generated by the material point. In other terms, the manifold created by a mass point does *not* possess a definite symmetry *of its own*. In sect.**5** we have mentioned an example given by Weyl [4]. Another example is Kerr's manifold: as we have seen, Kerr's oblate spheroid $r = m + (m^2 - a^2 \cos^2 \vartheta)^{1/2}$ can be transformed, with a simple collineation, into the sphere $r' = 2m$. In this way, *Kerr's manifold is "reduced" to Schwarzschild's manifold of a mass point at rest.* (*N.B.* – The use of a *Bildraum* for the proof of this "reduction" does not restrict the validity of our result).

The $\mathrm{d}s^2$ of eq. (1) can be considered, *e.g.*, as the limit of the $\mathrm{d}s^2$ of a homogeneous sphere of an incompressible fluid, whose radius goes to zero [6]. The $\mathrm{d}s^2$ of eq. (2) can be considered, *e.g.*, as the limit of the $\mathrm{d}s^2$ of the above contracting sphere as "viewed" by a frame which rotates with the angular velocity ω of eq. (2').

Summing up, the difference between Kerr's manifold and Schwarzschild's manifold is *only* a difference of reference systems: Kerr's metric is described by a *rotating* frame, Schwarzschild's metric by a *static* frame. In *both* cases the material agent is the *same*: a point mass.

7. – ***Aperçu***. – *i*) If the angular velocity of rotation contained in Kerr's metric is equal to zero, Kerr's manifold coincides with Schwarzschild's manifold created by a point mass at rest. – *ii*) The coordinate shift $r \rightarrow r + m + (m^2 - a^2 \cos^2 \vartheta)^{1/2}$ in Kerr's metric gives an expression of the $\mathrm{d}s^2$ with a *sole* (and "soft") singularity at $r = 0$. – *iii*) The oblate spheroid $r = m + (m^2 - a^2 \cos^2 \vartheta)^{1/2}$ can be transformed into the sphere $r' = 2m$. – *iv*) The difference between Kerr's metric and Schwarzschild's metric rests *only* on the difference between the respective reference frames. Accordingly, Kerr's metric can be "reduced" to Schwarzschild's metric by virtue of result

iii). – *v*) Of course, Kerr's potential g_{jk} gives origin to a Thirring-Lense *dragging* effect. However, all dragging forces are *only* caused by the chosen reference frames, and therefore do not have an *invariant* character. – *vi*) All *observational* data are in accord with our analysis. –

APPENDIX

We give here the banal proof that the vertical section of Kerr's surface $r = m + (m^2 - a^2 \cos^2 \vartheta)^{1/2}$, where $0 \leq r < +\infty$ and $0 \leq \vartheta \leq \pi$, is an ellipse.

If α, β are the semi-axes of a generic ellipse, and χ, $(0 \leq \chi < 2\pi)$, is Kepler's eccentric anomaly, our curve can be described by the following equations – as it is well known:

$$\xi = \alpha \cos \chi \quad ; \quad \eta = \beta \sin \chi \quad , \tag{A1}$$

where ξ, η are Cartesian orthogonal coordinates. If $\varrho^2 = \xi^2 + \eta^2$, we have:

$$\varrho = \left[\beta^2 - (\beta^2 - \alpha^2) \cos^2 \chi\right]^{1/2} \quad . \tag{A2}$$

Now, the equation $r - m = (m^2 - a^2 \cos^2 \vartheta)^{1/2}$ can also represent the half (ϑ is a colatitude) of the vertical section of the above Kerr's surface. We see that:

$$\varrho\,(\chi = 0) = \alpha \quad ; \quad \varrho\,(\chi = \pi/2) = \beta \quad , \tag{A3}$$

$$r\,(\vartheta = \pi/2) = 2m \quad ; \quad r\,(\vartheta = 0) = m + (m^2 - a^2)^{1/2} \quad , \tag{A4}$$

Q.e.d.

Accordingly, by virtue of the axial symmetry of Kerr's $\mathrm{d}s^2$, the surface $r = m + (m^2 - a^2 \cos^2 \vartheta)^{1/2}$ is a rotational ellipsoid. –

References

[1] R.H. Boyer and R.W Lindquist, *J. Math. Phys.*, **8** (1967) 265. And references therein.

[2] A. Loinger and T. Marsico, *arXiv:0809.122 v1* [physics.gen-ph] 7 Sep 2008. And references therein.

[3] M. Brillouin, *Journ. Phys. Rad.*, **23** (1923) 43; for an English translation by S. Antoci, see: *arXiv:physics/0002009* (February 3rd, 2000). See also the Schwarzschildian reference therein, which inspired Brillouin's paper. – Remark that in the new radial coordinate (eq. (3)), at $r = 0$ the angular velocity ω is different from zero.

[4] H. Weyl, *Ann. Physik*, **54** (1917) 117.
[5] See, *e.g.* L.P. Eisenhart, *Continuous Groups of Transformations* (Dover Publ., New York) 1961, p.208 *sqq.*.
[6] A. Loinger, *arXiv:gr-qc/9908009* (August 3rd, 1999).
[7] Cf., *e.g.* H. Weyl, *The Classical Groups - Their Invariants and their Representations*, (Princeton University Press, Princeton, NJ) 1946, p.112 *sqq.*; and G. Castelnuovo, *Lezioni di Geometria Analitica* (Albrighi, Segati e C., Milano, *etc.*) 1938, p.p. 494 and 540.

A.L. – Dipartimento di Fisica, Università di Milano, Via Celoria, 16 - 20133 Milano (Italy)

T.M. – Liceo Classico "G. Berchet", Via della Commenda, 26 - 20122 Milano (Italy)

E-mail address: angelo.loinger@mi.infn.it
E-mail address: martiz64@libero.it